The 6j-symbol: Recursion, Correlations and Asymptotics

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We study the asymptotic expansion of the {6j}-symbol using the Schulten-Gordon recursion relations. We focus on the particular case of the isosceles tetrahedron and we provide explicit formulas for up to the third order corrections beyond the leading order. Moreover, in the framework of spinfoam models for 3d quantum gravity, we show how these recursion relations can be used to derive Ward-Takahashi-like identities between the expectation values of graviton-like spinfoam correlations.

I. INTRODUCTION: {6J}-SYMBOL AND THE RECURSION FORMULA

Spinfoam models present a proposal for a well-defined path integral formulation for quantum gravity. They define the structure quantum space-time at the Planck scale. The current challenge is to study their semi-classical behavior at large scale, show that we recover general relativity and a locally flat space-time as expected and then to extract the perturbative quantum gravity correction to the classical gravitational dynamics. The main spinfoam models for 4d quantum gravity are the Barrett-Crane model [1] and the more recent family of models [2, 3, 4] based on coherent state techniques. These models are all derived from the reformulation of general relativity as a constrained topological BF-theory and attempt to define a discretization of the path integral over space-time geometries.

A recent proposal, which is actively investigated, to probe the semi-classical behavior of the spinfoam amplitudes is the test of the graviton propagator proposed by Rovelli and collaborators [5]. There have been several more or less rigorous analytical computations [6], as well as numerical simulations [7], showing that the most recent spinfoam models have the expected behavior reproducing at first order (and for the simplest space-time triangulation) Newton's law and the tensorial structure of the graviton. These rely on the computation of the leading order asymptotics of some spinfoam amplitudes for large spin. The next step is to go beyond the leading order and compute the quantum corrections which should correspond to the loop corrections of the standard perturbative quantum field theory approach. This necessarily requires a better understanding of the structure of the asymptotics of the spinfoam amplitudes. Here, we investigate this issue in the simplified framework of 3d quantum gravity.

In three space-time dimensions, gravity is a topological theory which can be exactly quantized as a spinfoam model, the well-known Ponzano-Regge model [8]. Its basic building block is the {6j}-symbol. Its asymptotics has been well-studied at leading order and been derived using various techniques: from the brute-force calculation based on the explicit formula of the {6j}-symbol in term of factorials [9] to more refined saddle point approximation on integral formulas [10]. Recently, these calculations have been pushed further in order to compute the corrections to the asymptotic behavior using both the saddle point technique [11] and the brute-force method [12]. These results apply to the computation of graviton-like correlations in 3d quantum gravity [11, 14]. This should lead to a better understanding of the structure of the quantum corrections of the spinfoam graviton propagator, which should be relevant to the four-dimensional case.

In the present paper, we are interested in the computation of the asymptotics of the {6j}-symbol through the use of recursion relations. As was first shown in [13], the {6j}-symbol satisfies a recursion formula which are intimately related to the topological invariance of the Ponzano-Regge spinfoam model. This formula turned out to be very useful: it can be approximated in the large spin limit by a second order differential equation and one can use it to derive the leading order of the asymptotics through a WKB approximation, but it also allows fast numerical calculations of the {6j}-symbol.

Here, we investigate two aspects of these recursion relations. First, we show how to extract the next-to-leading and subsequent corrections to the {6j}-symbol. We focus on the case of the isosceles tetrahedron (since this is the case relevant for the computation of graviton-like correlations [14]) and we compare our results with the previous work [11, 12]. Second, we study the consequences of the existence of such a recursion relation on the behavior of the graviton-like correlation functions in spinfoam models. We show that it leads to relations on these correlations functions, which relate the expectation values of different observables. These relations are similar to the Ward-Takahashi identities (and to the Schwinger-Dyson equation) in standard quantum field theory, which allow to relate

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different correlation functions of different orders. These equations play a key role in the study of the renormalization of quantum field theories, so we expect these new recursion relation to be equally relevant to the study of the renormalization/coarse-graining of spinfoam models.

II. THE RECURSION RELATION FOR THE ISOSCELES TETRAHEDRON

A. Exact and Approximate Recursion

We will focus on the isosceles tetrahedron, which is relevant for the computations of geometrical correlations in the simplest non-trivial toy model in 3d quantum gravity [14]. Such a tetrahedron has four of its edges of equal length with the two remaining opposite edges of arbitrary length. The corresponding isosceles {6j}-symbol is:

$$\{a,b\}_J \equiv \left\{ \begin{array}{ccc} a & J & J \\ b & J & J \end{array} \right\},$$

where $J \in \mathbb{N}/2$ and a, b are integers smaller than $2J$ (to satisfy the triangular inequality). The associated tetrahedron has edge lengths $l_j = d_j/2$ for $j = a, b, J$, where $d_j = 2j+1$ is the dimension of the SU(2) representation of spin j. The volume V of the tetrahedron is given by the simple formula:

$$V_J(a,b) = \frac{1}{12} l_a l_b \sqrt{4l_J^2 - (l_a^2 + l_b^2)}, \tag{1}$$

while the (exterior) dihedral angles θ_j can also be written in term of the edge lengths (see e.g. [11] for more details):

$$\cos\theta_a = -\frac{4l_J^2 - l_a^2 - 2l_b^2}{4l_J^2 - l_a^2}, \quad \cos\theta_b = -\frac{4l_J^2 - l_b^2 - 2l_a^2}{4l_J^2 - l_b^2}, \quad \cos\theta_J = \frac{-l_a l_b}{\sqrt{4l_J^2 - l_a^2}\sqrt{4l_J^2 - l_b^2}}. \tag{2}$$

The general recursion relation for the {6j}-symbol given by Schulten and Gordon in [13] simplifies in this specific isosceles case:

$$(l_a + \frac{1}{2}) \left[4l_J^2 - (l_a + \frac{1}{2})^2 \right] \{a+1,b\}_J - 2l_a \left[(4l_J^2 - l_a^2)\cos(\theta_a) + \frac{1}{4} \right] \{a,b\}_J + (l_a - \frac{1}{2}) \left[4l_J^2 - (l_a - \frac{1}{2})^2 \right] \{a-1,b\}_J = 0 \tag{3}$$

In the asymptotic regime, we know (analytically and numerically) the behavior of the {6j}-symbol at the leading order:

$$\{a,b\}_J \sim \{a,b\}_J^{LO} \equiv \frac{1}{\sqrt{12\pi V}} \cos\left(l_a\theta_a + l_b\theta_b + 4l_J\theta_J + \frac{\pi}{4} \right), \tag{4}$$

which is actually valid under the assumption that the tetrahedron with edge lengths l_a, l_b, l_J exists (else the generic asymptotics can be expressed in term of Airy functions). The oscillatory phase is given by the Regge action $S_R = l_a\theta_a + l_b\theta_b + 4l_J\theta_J$. Using the obvious trigonometric identity $\cos((n+1)\phi) + \cos((n-1)\phi) = 2\cos\phi\cos n\phi$, we can write an exact recursion relation for the leading order of the {6j}-symbol:

$$\sqrt{V_J(a+1,b)}\,\{a+1,b\}_J^{LO} - 2\cos\theta_a\,\sqrt{V_J(a,b)}\,\{a,b\}_J^{LO} + \sqrt{V_J(a-1,b)}\,\{a-1,b\}_J^{LO} = 0. \tag{5}$$

A similar recursion relation holds for b-shifts and also J-shifts.

The most natural idea is to compare this recursion relation for the leading order to the previous equation on the exact {6j}-symbol to see how to use them to extract the next-to-leading correction to the asymptotic behavior. We can first find the link between the leading order of equation (3) and the leading order of equation (5). Both equations can be written under the same form at the leading order:

$$\{a+1,b\}_J - 2\cos\theta_a\{a,b\}_J + \{a-1,b\}_J \approx 0, \tag{6}$$

which turns into a simple second order differential equation in the large spin limit. Then the next-to-leading order of the equation (3):

$$\begin{aligned} &\sqrt{V_J(a,b)}\left(1 + \tfrac{1}{2l_a}\left(1 - \tfrac{2l_a^2}{4l_J^2 - l_a^2}\right)\right)\ \{a+1,b\}_J - 2\cos\theta_a\sqrt{V_J(a,b)}\{a,b\}_J \\ &\qquad + \sqrt{V_J(a,b)}\left(1 - \tfrac{1}{2l_a}(1 - \tfrac{2l_a^2}{4l_J^2 - l_a^2})\right)\{a-1,b\}_J \approx 0 \end{aligned} \tag{7}$$

will have to be compared to an recursion relation for the next-to-leading order of the {6j}-symbol.

B. Pushing to the Next-to-Leading Order

We are interested in the asymptotic expansion of the {6j}-symbol. It was shown in previous works [11, 12] that l_j seems to be the right parameter to consider when studying the semi-classical behavior of the {6j}-symbol. So from now we write:

$$\left\{ \begin{array}{ccc} a & J & J \\ b & J & J \end{array} \right\} \equiv \{l_a, l_b\}_{l_J}.$$

Notice that shifting a by ± 1 is equivalent to shifting the edge length $l_a = a + 1/2$ by ± 1. We rescale now l_j by λl_j and we replace the exact {6j}-symbol by a series in $1/\lambda$ alternating cosines and sinus of the Regge action (shifted by $\pi/4$) in the previous equation (3). The fact that there is no mixing up of cosines and sinus at all order was show in [11]. More precisely, we write the {6j}-symbol asymptotic expansion under the form:

$$\begin{aligned} \{\lambda l_a, \lambda l_b\}_{\lambda l_J} = \frac{1}{\lambda^{3/2} D(l_a,l_b,l_J)} [\cos(\lambda S_R + \pi/4) + & \frac{F^{(1)}(l_a,l_b,l_J)}{\lambda} \sin(\lambda S_R + \pi/4) + \frac{G^{(1)}(l_a,l_b,l_J)}{\lambda} \cos(\lambda S_R + \pi/4)) \\ & + \frac{F^{(2)}(l_a,l_b,l_J)}{\lambda^2} \cos(\lambda S_R + \pi/4) + \frac{G^{(2)}(l_a,l_b,l_J)}{\lambda^2} \sin(\lambda S_R + \pi/4)) \\ & + \frac{F^{(3)}(l_a,l_b,l_J)}{\lambda^3} \sin(\lambda S_R + \pi/4) + \frac{G^{(3)}(l_a,l_b,l_J)}{\lambda^3} \cos(S_R + \pi/4) \\ & + \frac{F^{(4)}(l_a,l_b,l_J)}{\lambda^4} \cos(\lambda S_R + \pi/4) + \frac{G^{(4)}(l_a,l_b,l_J)}{\lambda^4} \sin(S_R + \pi/4) + O(\lambda^{-5})], \end{aligned} \tag{8}$$

where the pre-factor denominator $D(l_a, l_b, l_J)$ is given by the square-root of the tetrahedron volume as in equation (4). To study the asymptotics, it is convenient to factorize the whole equation (3) by $\lambda^{3/2}$. We then write $\{l_a \pm 1/\lambda, l_b\}_{l_J}$ for $\{\lambda l_a \pm 1, \lambda l_b\}_{\lambda l_J}$. We also factorize the coefficients of the recursion relation. We start by defining $C(l_j) = l_a(4l_J^2 - l_a^2) = \frac{16(A(la,l_J,l_J))^2}{l_a}$ where $A(a,b,c) = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}$ is the area of the triangle of edge lengths given by a, b and c. The coefficient which appears in front of $\{l_a \pm 1/\lambda, l_b\}_{l_J}$ becomes $C(l_a \pm 1/(2\lambda), l_b, l_J) = (l_a \pm 1/(2\lambda))(4l_J^2 - (l_a \pm 1/(2\lambda))$, where we underline that the shift is $\pm 1/(2\lambda)$ and not simply $\pm 1/\lambda$. We expand $C(l_a \pm 1/(2\lambda), l_b, l_J)$ in term of derivatives:

$$C(l_a \pm 1/(2\lambda), l_b, l_J) = \sum_n \frac{1}{n!} \frac{1}{(2\lambda)^n} \frac{\partial^n C}{\partial l_a^n}$$

with

$$\left\{ \begin{array}{l} C = l_a(4l_J^2 - l_a^2) \\ \frac{\partial C}{\partial l_a} = 4l_J^2 - 3l_a^2 \\ \frac{\partial^2 C}{\partial l_a^2} = -6l_a^2 \\ \frac{\partial^3 C}{\partial l_a^3} = -6 \\ \frac{\partial^n C}{\partial l_a^n} = 0 \text{ for } n \geq 4 \end{array} \right. \tag{9}$$

Then to express $\{l_a \pm 1/\lambda, l_b\}_{l_J}$ we need to expand $D(l_a \pm 1/\lambda)$, $F^{(i)}(l_a \pm 1/\lambda)$, and $G^{(i)}(l_a \pm 1/\lambda)$: $(i \in \{1 \cdots 4\})$

$$\left\{ \begin{array}{l} D(l_a \pm 1/\lambda) = D \pm \frac{1}{\lambda} \frac{\partial D}{\partial l_a} + \frac{1}{2\lambda^2} \frac{\partial^2 D}{\partial l_a^2} \pm \frac{1}{3!\lambda^3} \frac{\partial^3 D}{\partial l_a^3} + \frac{1}{4!\lambda^4} \frac{\partial^4 D}{\partial l_a^4} \\ F^{(i)}(l_a \pm 1/\lambda) = \sum_{k=0}^{4-i} (-1)^k \frac{1}{k!\lambda^k} \frac{\partial^k F^{(i)}}{\partial l_a^k} \\ G^{(i)}(l_a \pm 1/\lambda) = \sum_{k=0}^{4-i} (-1)^k \frac{1}{k!\lambda^k} \frac{\partial^k G^{(i)}}{\partial l_a^k} \end{array} \right. \tag{10}$$

$F^{(1)}(l_j)$ was computed in a previous paper [11, 12]. It was also suggested that the asymptotic expansion of the {6j}-symbol in term of the length scale λ is given by an alternative of cosines and sinus at each order, so we expect that $G^{(i)}(l_j) = 0$ for $\forall i \geq 1$. Finally, we also need to expand the Regge action $\lambda S_R(l_a \pm \frac{1}{\lambda})$, remembering that $\theta_j = \theta_j(l_a)$:

$$\lambda S_R(l_a \pm \frac{1}{\lambda}) = \lambda S_R + \sum_{k=0}^{4} \frac{(-1)^{k+1}}{(k+1)!\lambda^k} \frac{\partial^k \theta_a}{\partial l_a^k} \tag{11}$$

with

$$\begin{cases} \frac{\partial\theta_a}{\partial l_a} = \frac{-2l_a l_b}{(4l_J^2-l_a^2)\sqrt{4l_J^2-l_a^2-l_b^2}} \\ \\ \frac{\partial^2\theta_a}{\partial l_a^2} = -\frac{2l_b(4l_J^2l_a^2-2l_a^4-l_a^2l_b^2+16l_J^4-4l_b^2l_J^2)}{(4l_J^2-l_a^2)^2[4l_J^2-l_a^2-l_b^2]^{3/2}} \\ \\ \frac{\partial^3\theta_a}{\partial l_a^3} = -\frac{2l_al_b(24l_b^4l_J^2+40l_J^2l_a^2l_b^2-12l_J^2l_a^4+5l_a^4l_b^2-192l_J^4l_a^2-240l_J^4l_b^2+2l_a^2l_b^4+6l_a^6+576l_J^6)}{(4l_J^2-l_a^2)^3[4l_J^2-l_a^2-l_b^2]^{5/2}} \\ \\ \frac{\partial^4\theta_a}{\partial l_a^4} = \frac{1}{(4l_J^2-la^2)^4(4l_J^2-l_a^2-l_b^2)^{7/2}}(6(8l_a^{10}+8l_a^8l_b^2+152l_J^2l_a^6l_b^2-720l_J^4l_a^6+7l_a^6l_b^4+3520l_J^6l_a^4-1472l_J^4l_a^4l_b^2+2l_a^4l_b^6 \\ \quad +140l_a^4l_b^4l_J^2-560l_a^2l_b^4l_J^4-3840l_J^8l_a^2+2432l_J^6l_a^2l_b^2+48l_a^2l_J^2l_b^6+2048l_J^8l_b^2-448l_J^6l_b^4-3072l_J^{10}+32l_b^6l_J^4)l_b) \end{cases}$$

We can now write an asymptotic recursion equation from equations (3), (9), (8), (10) and (11) in terms of λ neglecting terms of order $O(\lambda^{-4})$ and smaller, assuming that λ is large. This leads to a couple of equations at each order, one for the cos-oscillations and one for the term in sin:

- The first equation is given by the terms of order λ^0 and it is trivially satisfied ($0=0$) since we have already written the leading order of the {6j}-symbol proportional to $\cos(S_R+\frac{\pi}{4})$ (the Ponzano-Regge asymptotic formulae).

- The second equation is given by the terms of order λ^{-1}:

$$\left(\frac{1}{2C}\frac{\partial C}{\partial l_a}-\frac{1}{D}\frac{\partial D}{\partial l_a}\right)\sin(\theta_a)+\frac{1}{2}\frac{\partial\theta_a}{\partial l_a}\cos(\theta_a)=0 \tag{12}$$

  which can be rewritten as a differential equation for D:

$$\frac{\partial \ln D}{\partial l_a}=\frac{1}{2}\left[\frac{\partial\theta_a}{\partial l_a}\frac{\cos\theta_a}{\sin\theta_a}+\frac{\partial \ln C}{\partial l_a}\right]. \tag{13}$$

  This allows to determine D: $\ln D=\frac{1}{2}\ln(C\sin(\theta_a))+K$, which simplifies into $D=K\sqrt{l_al_b\sqrt{4l_J^2-l_a^2-l_b^2}}$ where K is a constant factor. Thus this second equation shows that D is correctly proportional to the square-root of the volume V of the isosceles tetrahedron. To determine the normalization constant K (as well as $G^{(1)}$), the orthonormality property of {6j}-coefficients can be employed: $\sum_a 4l_a\sqrt{l_bl_{b'}}\{a,b\}_J\{a,b'\}_J=\delta_{bb'}$ and we get the $K=\sqrt{12\pi}$. The details are given in the next section.

- The third equation is given by the terms of order λ^{-2} and which are proportional to $\cos(S_R+\frac{\pi}{4})$

$$\begin{aligned}\frac{\partial F^{(1)}}{\partial l_a} = \frac{l_a}{4C\sin\theta_a}+ & \left(\frac{1}{2C}\frac{\partial C}{\partial l_a}-\frac{1}{D}\frac{\partial D}{\partial l_a}\right)\frac{1}{2}\frac{\partial\theta_a}{\partial l_a}+\frac{1}{6}\frac{\partial^2\theta_a}{\partial l_a^2} \\ & +\frac{\cos\theta_a}{\sin\theta_a}\left(\frac{1}{2D}\frac{\partial^2 D}{\partial l_a^2}-\frac{1}{8C}\frac{\partial^2 C}{\partial l_a^2}+\frac{1}{D}\frac{\partial D}{\partial l_a}\left(\frac{1}{2C}\frac{\partial C}{\partial l_a}-\frac{1}{D}\frac{\partial D}{\partial l_a}+(\frac{1}{2}\frac{\partial\theta_a}{\partial l_a})^2\right)\right)\end{aligned} \tag{14}$$

  where we used the fact that $\left(\frac{1}{2C}\frac{\partial C}{\partial l_a}-\frac{1}{D}\frac{\partial D}{\partial l_a}\right)\sin(\theta_a)+\frac{1}{2}\frac{\partial\theta_a}{\partial l_a}\cos(\theta_a)=0$ (eqn. (13)) to remove all the terms proportional to $F^{(1)}$ itself. The first term of the right-hand side of the equation (14) comes from the variation of the coefficient in front of $\{a,b\}_J$ in the recursion equation (3). The terms with a derivative of C with respect to l_a come from the coefficients in front of $\{a\pm1,b\}_J$ and $\{a,b\}_J$. The variation of C with respect to l_a is given by the variation of the areas of the triangles of the tetrahedron. From eqn.(13), we relate it to the variations of D (the volume) and to the variations of the dihedral angle θ_a: $\frac{1}{C}\frac{\partial C}{\partial l_a}=\frac{2}{D}\frac{\partial D}{\partial l_a}-\frac{\cos\theta_a}{\sin\theta_a}\frac{\partial\theta_a}{\partial l_a}$. The terms with a derivative of D with respect to l_a come from the variation of the leading order of the asymptotic of the {6j}-symbol and the terms with a derivative of the dihedral angle θ_a come from the variations of the Regge action S_R. We can now compute the derivative of $F^{(1)}$ with respect of l_a (equation (14)) in terms of l_a, l_b and l_J the edge lengths of the tetrahedron:

$$\begin{aligned}\frac{\partial F^{(1)}}{\partial l_a} = -\ & \frac{1}{48(l_a^2(-4l_J^2+l_a^2)^2(4l_J^2-l_a^2-l_b^2)^{(5/2)}l_b)}(-32l_b^6l_J^2l_a^2+10l_b^6l_a^4+96l_b^6l_J^4-960l_J^6l_b^4+15l_a^6l_b^4+400l_J^4l_b^4l_a^2-100l_J^2l_b^4l_a^4 \\ & -168l_J^2l_a^6l_b^2-1664l_J^6l_b^2l_a^2+20l_a^8l_b^2+576l_J^4l_b^2l_a^4+3072l_J^8l_b^2-3072l_J^{10}+48l_J^4l_a^6+2304l_J^8l_a^2-576l_J^6l_a^4)\end{aligned} \tag{15}$$

  and then easily integrate this equation over l_a:

$$\begin{aligned}F^{(1)}(l_j) &= \\ & -\frac{768l_J^6(l_J^2-l_a^2-l_b^2)+736l_J^4l_a^2l_b^2+240l_J^4(l_a^4+l_b^4)-176l_J^2l_a^2l_b^2(l_a^2+l_b^2)-24l_J^2(l_a^6+l_b^6)+10l_a^2l_b^2(l_a^4+l_b^4)+25l_a^4l_b^4}{24\left(4l_J^2-l_b^2\right)\left(2l_J^2-l_a^2\right)\left(4l_J^2-l_b^2-l_a^2\right)^{3/2}l_al_b}+Z(l_b,l_J)\end{aligned} \tag{16}$$

The integration constant $Z(l_b, l_J)$ can be determined using the symmetry properties of the {6j}-symbol: symmetry of the isosceles {6j}-symbol with respect to l_a and l_b, coupling of l_a, l_b and l_J by this isosceles {6j}-symbol and homogeneity of $F^{(1)}$ ($[F^{(1)}] = l_j^{-1}$) imply that $Z(l_b, l_J) = 0$. Then this gives us the same result as in the previous paper [12]. Moreover, using the definitions of the tetrahedron volume (1) and of the dihedral angles (2), we can express $F^{(1)}$ in terms of some geometrical characteristics of the tetrahedron:

$$F^{(1)} = -\frac{\cos\theta_J \left(3(12V)^8 - (12V)^4 l_a^4 l_b^4 \left(3(l_a^2 - l_b)^2 + 2l_a^2 l_b^2\right) - l_a^{12} l_b^{12}\right) + 6l_a^{12} l_b^{12}}{48(12V)^3 l_a^8 l_b^8} \tag{17}$$

- The fourth equation is given by the terms of order λ^{-2} and which are proportional to $\sin(S_R + \frac{\pi}{4})$. It is the same equation as the previous one for $G^{(1)}$ but the right-hand side is now equal to zero (homogenous equation). That is we simply get that

$$\frac{\partial G^{(1)}}{\partial l_a} = 0 \tag{18}$$

so $G^{(1)} = Z(l_b, l_J)$ is just a constant of integration. Once again the symmetry properties of the {6j}-symbol implies that $G^{(1)} = 0$.

- The next equation is given by the terms of order λ^{-3} and which are proportional to $\sin(S_R + \frac{\pi}{4})$. We get an equation for the first derivative of $F^{(2)}$ with respect to l_a

$$\begin{aligned}
\frac{\partial F^{(2)}(l_j)}{\partial l_a} &= \frac{\cos\theta_a}{2\sin\theta_a}\frac{\partial^2 F^{(1)}}{\partial l_a^2} - \left(\frac{1}{\sin^2\theta_a}\frac{\partial\theta_a}{\partial l_a} + F^{(1)}\right)\frac{\partial F^{(1)}}{\partial l_a} \\
&+ \frac{\cos\theta_a}{\sin\theta_a}\left[-\frac{1}{4!}\frac{\partial^3\theta_a}{\partial l_a^3} + \left(\frac{1}{D}\frac{\partial D}{\partial l_a}\frac{1}{2C}\frac{\partial C}{\partial l_a} + \frac{1}{2D}\frac{\partial^2 D}{\partial l_a^2} - \left(\frac{1}{D}\frac{\partial D}{\partial l_a}\right)^2 - \frac{1}{8C}\frac{\partial^2 C}{\partial l_a^2} + \frac{1}{3!}\left(\frac{1}{2}\frac{\partial\theta_a}{\partial l_a}\right)^2\right)\frac{1}{2}\frac{\partial\theta_a}{\partial l_a} + \left(\frac{1}{D}\frac{\partial D}{\partial l_a} - \frac{1}{2C}\frac{\partial C}{\partial l_a}\right)\frac{1}{3!}\frac{\partial^2\theta_a}{\partial l_a^2}\right] \\
&+ \frac{1}{2C}\frac{\partial C}{\partial l_a}\frac{1}{2D}\frac{\partial^2 D}{\partial l_a^2} + \frac{1}{8C}\frac{\partial^2 C}{\partial l_a^2}\frac{1}{D}\frac{\partial D}{\partial l_a} + \left(\frac{1}{2C}\frac{\partial C}{\partial l_a} - \frac{1}{D}\frac{\partial D}{\partial l_a}\right)\frac{1}{2}\left(\frac{1}{2}\frac{\partial\theta_a}{\partial l_a}\right)^2 + \frac{1}{3!}\frac{\partial^2\theta_a}{\partial l_a^2}\frac{1}{2}\frac{\partial\theta_a}{\partial l_a} + \left(\frac{1}{D}\frac{\partial D}{\partial l_a}\right)^3 - \frac{1}{D}\frac{\partial D}{\partial l_a}\frac{1}{D}\frac{\partial^2 D}{\partial l_a^2} + \frac{1}{3!D}\frac{\partial^3 D}{\partial l_a^3} \\
&- \frac{1}{8\cdot 3!C}\frac{\partial^3 C}{\partial l_a^3} - \frac{1}{2C}\frac{\partial C}{\partial l_a}\left(\frac{1}{D}\frac{\partial D}{\partial l_a}\right)^2
\end{aligned} \tag{19}$$

We recall that D is proportional to the square root of the tetrahedron volume, C can be expressed in terms of the volume V and the sinus of the dihedral angle θ_a (see equation (13)). To integrate this equation, we first express explicitly [1]it in terms of l_a, l_b and l_J, and then deduce $F^{(2)}$:

$$\begin{aligned}
F^{(2)}(l_j) &= \frac{-1}{4608\left((4l_J^2-l_a^2)^2(4l_J^2-l_b^2)^2(4l_J^2-l_a^2-l_b^2)^3 l_a^2 l_b^2\right)}(-2359296 l_a^2 l_J^{10} l_b^4 - 224512 l_a^6 l_J^6 l_b^4 + 100 l_a^{12} l_b^4 + 576 l_J^4 l_b^{12} + 112896 l_J^8 l_b^8 \\
&+2727936 l_a^4 l_J^{12} + 5308416 l_J^{16} + 212 l_b^{10} l_a^6 - 5898240 l_a^2 l_J^{14} - 11520 l_a^{10} l_J^6 + 941056 l_a^4 l_J^8 l_b^4 + 31584 l_a^8 l_J^4 l_b^4 \\
&-2416 l_a^4 l_b^{10} l_J^2 - 79872 l_a^8 l_J^6 l_b^2 - 480 l_b^{12} l_J^2 l_a^2 - 7040 l_a^8 l_b^6 l_J^2 - 2416 l_a^{10} l_J^2 l_b^4 + 100 l_a^4 l_b^{12} + 212 l_a^{10} l_b^6 \\
&+2727936 l_J^{12} l_b^4 - 700416 l_J^{10} l_b^6 - 5898240 l_J^{14} l_b^2 - 11520 l_J^6 l_b^{10} - 700416 l_a^6 l_J^{10} + 609 l_a^8 l_b^8 + 112896 l_a^8 l_J^8 \\
&+576 l_a^{12} l_J^4 - 2359296 l_a^4 l_J^{10} l_b^2 + 528384 l_a^6 l_J^8 l_b^2 + 5849088 l_a^2 l_J^{12} l_b^2 - 79872 l_a^2 l_b^8 l_J^6 + 31584 l_a^4 l_b^8 l_J^4 - 7040 l_a^6 l_b^8 l_J^2 \\
&-224512 l_a^4 l_b^6 l_J^6 + 58816 l_a^6 l_b^6 l_J^4 + 8640 l_a^{10} l_J^4 l_b^2 + 8640 l_b^{10} l_J^4 l_a^2 - 480 l_a^{12} l_J^2 l_b^2 + 528384 l_a^2 l_J^8 l_b^6)
\end{aligned} \tag{20}$$

which is the only result with the required symmetries[2]. The geometrical meaning of this function does not seem obvious. Nevertheless, we can give a more compact expression for the denominator of $F^{(2)}$:

$$(4l_J^2 - l_a^2)^2(4l_J^2 - l_b^2)^2(4l_J^2 - l_a^2 - l_b^2)^3 l_a^2 l_b^2 = \frac{(12V)^6}{\cos^4\theta_J}. \tag{21}$$

---

[1] $\frac{\partial F^{(2)}}{\partial l_a}(l_j) = -\frac{1}{2304\left((4l_J^2-l_b^2)l_a^3(4l_J^2-l_a^2)^3(4l_J^2-l_a^2+l_b^2)^4 l_b^2\right)}(-1604 l_a^8 l_b^8 l_J^2 + 1250816 l_a^4 l_J^8 l_b^6 - 207104 l_a^6 l_b^6 l_J^6 + 31904 l_a^{10} l_J^4 l_b^4 - 169344 l_a^4 l_b^8 l_J^6 - 3920 l_a^6 l_b^{10} l_J^2 + 24992 l_a^6 l_b^8 l_J^4 - 46848 l_a^{10} l_J^6 l_b^2 - 7129088 l_a^4 l_J^{10} l_b^4 + 1770496 l_a^6 l_J^8 l_b^4 + 16832 l_a^4 l_b^{10} l_J^4 + 34368 l_a^8 l_J^4 l_b^6 - 6816 l_a^{12} l_J^4 l_b^2 - 278912 l_a^8 l_J^6 l_b^4 + 14524416 l_a^2 l_J^{12} l_b^4 + 486144 l_a^2 l_J^8 l_b^8 - 560 l_b^{12} l_a^4 l_J^2 - 43776 l_a^2 l_b^{10} l_J^6 - 3317760 la^2 lJ^{10} l_b^6 + 22241280 l_a^4 l_J^{12} l_b^2 + 794 l_a^{12} l_b^6 - 6955008 l_a^6 l_J^{10} l_b^2 + 2801664 l_J^{12} l_b^6 + 672 l_a^{14} l_b^2 l_J^2 - 26542080 l_a^4 l_J^{14} - 451584 l_J^{10} l_b^8 + 46080 l_J^8 l_b^{10} - 2304 l_b^{12} l_J^6 - 21233664 l_J^{18} + 37158912 l_a^2 l_J^{16} + 1072128 l_a^8 l_J^8 l_b^2 - 1528 la^{12} l_b^4 l_J^2 - 10911744 l_J^{14} l_b^4 - 35979264 l_a^2 l_J^{14} l_b^2 + 1728 l_b^{12} l_J^4 l_a^2 + 9953280 l_a^6 l_J^{12} + 228096 l_a^{10} l_J^8 - 10368 l_a^{12} l_J^6 - 2073600 l_a^8 l_J^1 0 + 27 l_a^{10} l_b^8 + 23592960 l_J^{16} l_b^2 + 400 l_a^8 l_b^{10} - 88 l_a^{14} l_b^4 - 8144 l_a^{10} l_b^6 l_J^2 + 100 l_b^{12} l_a^6)$

[2] If the result is not symmetric after integration, a non-null integration constant has to be added and its determination can be done using the symmetry properties of the {6j}-symbol. Indeed, we have $\frac{\partial F^{(2)}}{\partial l_a} = H(l_a, l_b, l_J)$ so by integration over l_a, $F^{(2)}(l_j) = h(l_a, l_b, l_J) + Z(l_b, l_J)$. Moreover by symmetry, we must have $\frac{\partial F^{(2)}}{\partial l_b} = H(l_a = l_b, l_b = l_a, l_J)$ and then integrating over l_b, we obtain a second expression for $F^{(2)}$: $F^{(2)}(l_j) = h(l_a = l_b, l_b = l_a, l_J) + Z(l_a, l_J)$ which implies that the constant of integration satisfies $Z(l_b, l_J) - Z(l_a, l_J) = h(l_a = l_b, l_b = l_a, l_J) - h(l_a, l_b, l_J)$. This equation allows to determine Z and to get (20).

- The next equation comes from the terms of order λ^{-3} which are proportional to $\cos(S_R+\frac{\pi}{4})$:

$$\frac{\partial G^{(2)}}{\partial l_a}(l_j)=0 \tag{22}$$

which implies once again that $G^{(2)}=Z(l_b,l_J)$ is a constant of integration. Then the symmetry properties of the {6j}-symbol implies $G^{(2)}(l_j)=0$.

We can now give the asymptotic expansion of an isosceles {6j}-symbol until the next to next to leading order (NNLO):

$$\{l_a,l_b\}^{\mathrm{NNLO}}_{l_J}=\frac{1}{\sqrt{12\pi V_{l_J}(l_a,l_a)}}\left[\cos(S_R+\frac{\pi}{4})+F^{(1)}(l_j)\sin(S_R+\frac{\pi}{4})+F^{(2)}(l_j)\cos(S_R+\frac{\pi}{4})\right] \tag{23}$$

where the expression for $F^{(1)}$ and $F^{(2)}$ are given by equations (17) and (20). This result seems to confirm that the expansion of the {6j}-symbol is a series alternating cosines and sinus of the Regge action (shift by $\frac{\pi}{4}$). In the case of an equilateral tetrahedron, all the edges have the same length, that is $l_a=l_b=l_J=l$ and $V=\frac{\sqrt{2}}{12}l^3$. Then equation (23) reduces to:

$$\{6j\}^{\mathrm{NNLO}}_{\mathrm{equi}}=\frac{1}{\sqrt{\pi l^3\sqrt{2}}}\cos(S_R+\frac{\pi}{4})-\frac{31}{72\,2^{1/4}\,2^{5/2}\sqrt{\pi l^5}}\sin(S_R+\frac{\pi}{4})-\frac{45673}{20736}\frac{1}{2^{1/4}\,2^4\sqrt{\pi l^7}}\cos(S_R+\frac{\pi}{4}) \tag{24}$$

where the Regge action is given by $S_R=6l\theta$ and $\theta=\theta_a=\theta_b=\theta_J=\arccos(-1/3)$. This result is confirmed by numerical simulations. The plot figure 1 represents numerical simulations of the equilateral {6j}-symbol minus its approximation (24). Moreover, to enhance the comparison, we have multiplied by $l^{7/2}$ to see how the coefficient of the NNLO is approached and we have divided by $\sin(S_R+\frac{\pi}{4})$ (oscillations of the next to next to next to leading order) to suppress the oscillations. This gives an error that decreases as expected as l^{-1}.

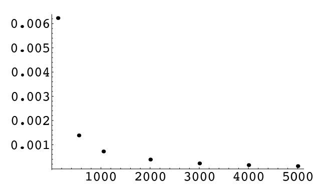

FIG. 1: Difference between the equilateral {6j}-symbol and the analytical result (24). The x-axis stands for $l=d/2$ and d goes from 100 to 5000. The error decreases as expected as l^{-1} confirming our asymptotic formula.

- The next two equations come from the terms of order λ^{-4}. The equation for $G^{(3)}$ is the same as the one for $G^{(1)}$ and $G^{(2)}$, that is: $\frac{\partial G^{(3)}}{\partial l_a}=0$. Using the same arguments of symmetry we deduce that $G^{(3)}=0$. This confirms our expectation of a series alternating cosines and sines in the asymptotic of the {6j}-symbol:

$$\{\lambda l_a,\lambda l_b\}_{\lambda l_J}=\frac{1}{\lambda^{3/2}D(l_a,l_b,l_J)}\left[\cos(\lambda S_R+\frac{\pi}{4})+\sum_{k=1}^{\infty}\frac{F^{(k)}(l_a,l_b,l_J)}{\lambda^k}\cos(\lambda S_R+\frac{\pi}{4}+\epsilon(k)\frac{\pi}{2})\right], \tag{25}$$

where $\epsilon(k)=-1$ when k is odd and $\epsilon(k)=0$ when k is even. We already have an expression for $F^{(1)}$ and $F^{(2)}$. The equation for $F^{(3)}$ is the second equation of order λ^{-4} and gives its first derivative with respect to l_a in terms of l_a, l_b and l_J. It is straightforward (though lengthy) to integrate it over l_a and get the expression[3]of $F^{(3)}$ in terms of l_a, l_b and l_J. In the equilateral case ($l_a=l_b=l_J=l$), the formula reduces to:

$$F^{(3)}=\frac{28833535}{17915904}\frac{1}{2^{9/2}2^{1/4}l^3} \tag{26}$$

Therefore, we have the expression of the asymptotic expansion of the equilateral {6j}-symbol up to the next-to-next-to-next to leading order (NNNLO):

$$\{6j\}^{\mathrm{NNNLO}}_{\mathrm{equi}} = \frac{1}{2^{1/4}\sqrt{\pi l^3}} \left[\cos(S_R+\frac{\pi}{4}) - \frac{31}{72\,2^{5/2}\,l}\sin(S_R+\frac{\pi}{4}) - \frac{45673}{20736\,2^4\,l^2}\cos(S_R+\frac{\pi}{4}) + \frac{28833535}{17915904\,2^{9/2}\,l^3}\sin(S_R+\frac{\pi}{4})\right]. \tag{27}$$

We check this result numerically by computing it using Mathematica for two values of spins $j = 50$ and $j = 100$. More precisely, we computed the renormalized error $\frac{(\{6j\}_{\mathrm{equi}} - \{6j\}^{\mathrm{NNNLO}}_{\mathrm{equi}})l^{9/2}}{\cos(S_R+\frac{\pi}{4})}$ and we got the expected $1/\lambda$-behavior. However for $l > 100$, Mathematica is not accurate enough and the numerical errors are too important to get exploitable results. In the general isosceles case, the expression of $F^{(3)}$ is quite complicated and its geometrical interpretation remains to be understood. Nevertheless, we can again give as before a more compact formula for the denominator of $F^{(3)}$:

$$\mathrm{denominator}_{F^{(3)}} = 3317760(4l_J^2 - l_a^2 - l_b^2)^{9/2} l_a^3 l_b^3 (4l_j^2 - l_a^2)^3 (4l_J^2 - l_b^2)^3 = 30(48)^3 \frac{(12V_J(a,b))^9}{\cos^6\theta_J} \tag{28}$$

From this equation, the equation giving the denominator of $F^{(2)}$ and remembering that the denominator of $F^{(1)}$ can be written under a similar form: $\mathrm{denominator}_{F^{(1)}} = 48\frac{(12V)^3}{\cos^2\theta_J}$, we can conjecture that:

$$\mathrm{denominator}_{F^{(k)}} \propto \frac{(12V_J(a,b))^{3k}}{(\cos\theta_J)^{2k}} \tag{29}$$

where $F^{(k)}$ are the terms appearing in the asymptotic expansion of the {6j}-symbol (25). And consequently, the numerator of $F^{(k)}$ is a polynomial in l_j of degree $8k$.

So, using the recursion relation for the isosceles {6j}-symbol as well as its symmetry properties, we have computed explicitly the asymptotic expansion of the isosceles {6j}-symbol to the fourth order up to an overall factor K (this integration constant K comes from the integration of the first equation (13)). The well-known value $K = \sqrt{12\pi}$ which already appears in the Ponzano-Regge formula can be obtained easily using the unitary property of the {6j}-symbol, as we show in the next section. The equilateral case has been checked against numerical calculations. This method using the recursion relation is fairly easy to implement. It requires integrating a rational fraction at each level and does not involve neither Riemann sum nor saddle point analysis. Moreover, since the coefficient C of the recursion relation (3) is a polynomial of degree 3; $\frac{\partial^n C}{\partial l_a^n} = 0$ for $n \geq 4$. Therefore, we expect to get a stable relation for the first derivative of $F^{(k)}$ and $G^{(k)}$ with respect to l_a for $k \geq 3$. On one hand, this allows to prove that $G^{(k)}$ always vanishes; and on the other hand, it should provide a systematic method to extract $F^{(k)}$ for arbitrary order k.

We conclude this section with a general remark on the asymptotic expansion of the {6j}-symbol. In the context of 3d quantum gravity, it is often argued that the leading order of the {6j}-symbol is a $\cos(S)$ instead of a complex phase $\exp(+iS)$, thus reflecting that the path integral is invariant under a change of (local) orientation (see e.g.[15]). This obviously neglects the $+\pi/4$ shifts, which can be considered as a quantum effect (like an ordering ambiguity). However, in the light of the present expansion, it is clear that we have terms of the type $\sin(S)$ beyond the leading order and such terms are not invariant under the change $S \to -S$. This means that the role of this symmetry in the spinfoam path integral should be more subtle than originally thought.

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[3] $F^{(3)}(l_j) = -\frac{1}{3317760(l_a^3(l_a-2l_J)(l_a+2l_J)(4l_J^2-l_b^2)^3(4l_J^2-l_a^2-l_b^2)^{(9/2)}l_b^3(4l_J^2-l_a^2)^2)}(-6965741568l_a^6l_J^{10}l_b^8 + 1069728l_a^6l_b^{16}l_J^2 + 28270080l_a^{16}l_J^6l_b^2 - 743040l_J^4l_a^2l_b^{18} + 1750503260160l_J^{20}l_a^2l_b^2 + 379404800l_a^{12}l_J^6l_b^6 + 788705280l_a^{10}l_J^{10}l_b^4 - 33547200l_a^6l_b^{14}l_J^4 - 98578608l_a^{12}l_b^8l_J^4 - 81032970240l_a^8l_J^{14}l_b^2 + 379404800l_a^6l_b^{12}l_J^6 - 389214720l_J^8l_a^2l_b^{14} - 323813376000l_a^6l_J^{18} - 156036l_a^8l_b^{16} - 22176l_J^2l_a^4l_b^{18} - 3824640l_a^{16}l_J^4l_b^4 - 28408l_a^{18}l_b^6 - 31104000l_a^{16}l_J^8 - 1262270545920l_J^{18}l_a^2l_b^4 - 1262270545920l_J^{18}l_a^4l_b^2 + 461814497280l_J^{16}l_a^2l_b^6 + 1069728l_a^{16}l_J^2l_b^6 + 74144841728l_a^6l_J^{12}l_b^6 + 824520278016l_J^{16}l_a^4l_b^4 + 461814497280l_a^6l_J^{16}l_b^2 + 788705280l_J^{10}l_a^4l_b^{10} - 267541217280l_a^6l_J^{14}l_b^4 + 115174656l_J^6l_a^4l_b^{14} - 999532800l_J^8l_a^4l_b^{12} + 68611276800l_a^8l_J^{16} - 3824640l_J^4l_a^4l_b^{16} - 1242078720l_a^{10}l_J^8l_b^6 + 28270080l_J^6l_a^2l_b^{16} + 38483472384l_J^{12}l_a^4l_b^8 + 2157096960l_a^{12}l_J^{10}l_b^2 - 1242078720l_a^6l_b^{10}l_J^8 + 509607936000l_J^{24} - 33547200l_a^{14}l_b^6l_J^4 + 1222041600l_J^{12}l_a^2l_b^{10} + 11119152l_a^{12}l_b^{10}l_J^2 - 999532800l_a^{12}l_J^8l_b^4 + 783522201600l_J^{20}l_a^4 + 2157096960l_J^{10}l_a^2l_b^{12} - 28408l_a^6l_b^{18} - 389214720l_a^{14}l_J^8l_b^2 - 783601920l_a^8l_J^8l_b^8 - 98578608l_a^8l_J^4l_b^{12} - 6965741568l_a^8l_J^{10}l_b^6 - 323813376000l_J^{18}l_b^6 + 1222041600l_a^{10}l_J^{12}l_b^2 + 115174656l_a^{14}l_J^6l_b^4 - 743040l_a^{18}l_J^4l_b^2 - 22176l_a^{18}l_J^2l_b^4 - 81032970240l_J^{14}l_a^2l_b^8 + 7201368l_a^{14}l_b^8l_J^2 - 114011616l_a^{10}l_b^{10}l_J^4 - 985242009600l_J^{22}l_b^2 + 568565760l_a^{10}l_b^8l_J^6 - 1401753600l_a^{12}l_J^{12} - 267541217280l_J^{14}l_a^4l_b^6 + 568565760l_a^8l_J^6l_b^{10} + 11119152l_a^{10}l_b^{12}l_J^2 + 783522201600l_J^20l_b^4 - 342523l_a^{12}l_b^{12} + 38483472384l_a^8l_J^{12}l_b^4 - 3981312000l_a^{10}l_J^{14} - 156036l_a^{16}l_b^8 - 511602l_a^{10}l_b^{14} + 344217600l_b^{14}l_J^{10} - 1401753600l_b^{12}l_J^{12} + 1036800l_b^{18}l_J^6 + 344217600l_a^{14}l_J^{10} + 68611276800l_J^{16}l_b^8 - 3981312000l_J^{14}l_b^{10} - 511602l_a^{14}l_b^{10} - 31104000l_b^{16}l_J^8 - 985242009600l_J^{22}l_a^2 + 1036800l_a^{18}l_J^6 + 7201368l_a^8l_b^{14}l_J^2)$

C. Consequences of the unitary property of the {6j}-symbols

The orthogonality property of the {6j}-symbols states that:

$$\sum_{l_a} 4l_a\sqrt{l_b l_{b'}}\{l_a,l_b\}_{l_J}\{l_a,l_{b'}\}_{l_J} = \delta_{l_b l_{b'}} \tag{30}$$

This relation corresponds to the unitarity of the evolution in the Ponzano-Regge 3d quantum gravity. We want to use this property to determine the constant of the leading order of the {6j}-symbol. From the recursion relation we have shown that $\{l_a,l_b\}_{l_J}^{\mathrm{LO}} = \frac{K}{\sqrt{V_J(a,b)}}\cos(S_R+\frac{\pi}{4})$; but K is still undetermined. For large spin and for $l_b \approx l_{b'}$, we can approximate the unitary property at the leading order in $(l_b - l_{b'})$ by:

$$\int_0^\infty dl_a 4l_a l_b \frac{K^2}{V_J(a,b)}\cos(S_R(l_a,l_b)+\frac{\pi}{4})\cos(S_R(l_a,l_{b'})+\frac{\pi}{4}) \approx \delta(l_b-l_{b'}). \tag{31}$$

The product of the cosines can be simplified at leading order:

$$\begin{aligned}\cos(S_R(l_a,l_b)+\frac{\pi}{4})\cos(S_R(l_a,l_{b'})+\frac{\pi}{4}) &= \frac{1}{2}\left[\cos(S_R(l_a,l_b)+S_R(l_a,l_{b'})+\frac{\pi}{2})+\cos(S_R(l_a,l_b)-S_R(l_a,l_{b'}))\right]\\ &\sim \frac{1}{2}\left[\cos(2S_R(l_a,l_b)+\frac{\pi}{2})+\cos((l_b-l_{b'})\theta_b)\right],\end{aligned}$$

where the dihedral angle $\theta_b = \arccos\left(-\frac{4l_J^2-l_b^2-2l_a^2}{4l_J^2-l_b^2}\right)$ is considered as a function of the length l_a. We do a saddle point approximation. The first term oscillates and its integral is exponentially suppressed. And we are left with the second term, which should satisfy the following equation:

$$\int_{-\infty}^\infty dl_a l_a l_b \frac{K^2}{V_J(a,b)}\cos((l_b-l_{b'})\theta_b) \approx \delta(l_b-l_{b'}) \tag{32}$$

We recall that:

$$\frac{1}{2\pi}\int_{-\infty}^\infty dl_a \cos(l_a(l_b-l_{b'})) = \delta(l_b-l_{b'})$$

therefore we can conclude that

$$l_a l_b \frac{K^2}{V} = \frac{1}{2\pi}\left|\frac{\partial\theta_b}{\partial l_a}\right|. \tag{33}$$

θ_b and l_b are so conjugate variables and K comes from the Jacobian of the change of variables between l_a and θ_b. Computing the derivative of the dihedral angle gives:

$$\frac{\partial\theta_b}{\partial l_a} = \frac{-2}{\sqrt{4l_J^2-l_a^2-l_b^2}} = \frac{-l_a l_b}{6V_J(a,b)} \quad\Rightarrow\quad K=\frac{1}{\sqrt{12\pi}}. \tag{34}$$

Moreover, pushing the approximation of the unitary property to the next to leading order in $(l_b - l_{b'})$ and using the next to leading order of the {6j}-symbol shows that $G^{(1)} = 0$. This was already shown in the previous part using the recursion relation and the symmetry properties of the {6j}-symbol and comes as a confirmation.

III. "WARD-TAKAHASHI IDENTITIES" FOR THE SPINFOAM GRAVITON PROPAGATOR

We are interested in the two-point function in 3d quantum gravity for the simplest triangulation given by a single tetrahedron. This provides the first order of the "spinfoam graviton propagator" in 3d quantum gravity.

Considering the isosceles tetrahedron, we focus on the correlations between the two representations a and b:

$$\langle\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_{\psi_J} = \frac{1}{Z}\sum_{a,b}\psi_J(a)\psi_J(b)\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\{a,b\}_J, \qquad Z \equiv \sum_{a,b}\psi_J(a)\psi_J(b)\{a,b\}_J, \tag{35}$$

where $\psi_J(j)$ is the boundary state, which depends also on the bulk length scale J, and $\mathcal{O},\widetilde{\mathcal{O}}$ are the observables whose correlation we are studying.

Now, inserting a recursion relation with shifts on a, b or J in the sum over the representation labels $\sum_{a,b}$ leads to equations relating the expectation values of different observables. We distinguish two cases: when the state ψ_J does not change or when the length scale J also varies.

A. Relating Observables

Inserting the recursion relation on a-shifts in the definition of the correlation function, we obtain the following exact identity:

$$\begin{aligned}\langle\tfrac{\psi_J(a-1)}{\psi_J(a)}\mathcal{O}(a-1)\widetilde{\mathcal{O}}(b)(l_a-\tfrac{1}{2})(4l_J^2-(l_a-\tfrac{1}{2})^2)\rangle_\psi\ &-\langle\mathcal{O}(a)\widetilde{\mathcal{O}}(b)2l_a(2\cos\theta_a(4l_J^2-l_a^2)+\tfrac{1}{4})\rangle_\psi\\ &+\langle\tfrac{\psi_J(a+1)}{\psi_J(a)}\mathcal{O}(a+1)\widetilde{\mathcal{O}}(b)(l_a+\tfrac{1}{2})(4l_J^2-(l_a+\tfrac{1}{2})^2)\rangle_\psi=0.\end{aligned}\tag{36}$$

We call this a Ward identity for our spinfoam correlation. If the observable diverges at $a=0$, more precisely if it contains terms in $1/a$ or in $1/(a+1)$, then we need to take into account extra boundary terms in this equation corresponding to contributions at $a=0$. But all observables usually considered are regular in this sense.

Then one can choose different sets of observables $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ and one gets different identities on the correlation functions of the spinfoam model. For example, taking $\mathcal{O}(a)=l_a$, we get:

$$\begin{aligned}\langle\tfrac{\psi_J(a-1)}{\psi_J(a)}\widetilde{\mathcal{O}}(b)(l_a-1)(l_a-1/2)(4l_J^2-(l_a-1/2)^2)\rangle_\psi\ &-\langle\widetilde{\mathcal{O}}(b)(2\cos\theta_a l_a^2(4l_J^2-l_a^2)+l_a^2/2)\rangle_\psi\\ &+\langle\tfrac{\psi_J(a+1)}{\psi_J(a)}\widetilde{\mathcal{O}}(b)(l_a+1)(l_a+1/2)(4l_J^2-(l_a+1/2)^2)\rangle_\psi=0.\end{aligned}$$

We recall that the area of the triangle of edge lengths given by $l_a,\ l_J,\ l_J$ is equal to $A(l_a,l_J)=\frac{1}{4}l_a\sqrt{4l_J^2-l_a^2}$; then $(l_a\pm1)(l_a\pm1/2)(4l_J^2-(l_a\pm1/2)^2)=16[A^2(l_a\pm1/2,l_J)\pm\frac{A^2(l_a\pm1/2,l_J)}{2(l_a\pm1/2)}]$, therefore we can rewrite the previous equation as an equation between correlation functions of the observable $\widetilde{\mathcal{O}}(b)$ and different observables proportional to the square of the triangle area $A(l_a,l_J)$:

$$\begin{aligned}\langle\tfrac{\psi_J(a-1)}{\psi_J(a)}[A^2(l_a-1/2,l_J)-\tfrac{A^2(l_a-1/2,l_J)}{2(l_a-1/2)}]\widetilde{\mathcal{O}}(b)\rangle_\psi\ &-\langle(2\cos\theta_a A^2(l_a,l_J)+l_a^2/2)\widetilde{\mathcal{O}}(b)\rangle_\psi\\ &+\langle\tfrac{\psi_J(a+1)}{\psi_J(a)}[A^2(l_a+1/2,l_J)+\tfrac{A^2(l_a+1/2,l_J)}{2(l_a+1/2)}]\widetilde{\mathcal{O}}(b)\rangle_\psi=0.\end{aligned}$$

The standard choice of boundary is a phased Gaussian [5, 14, 16]:

$$\psi_J(j)\ \sim\ e^{i2l_j\vartheta}e^{-2\alpha\frac{(l_j-l_J)^2}{l_J}},\tag{37}$$

where ϑ is a fixed angle defining a posteriori the external curvature of the boundary and α is an arbitrary real positive number (which can be fixed by the requirement of a physical state [16]). In this case, we can compute explicitly the ratios $\psi(a\pm1)/\psi(a)$ entering the Ward identity:

$$\frac{\psi_J(a\pm1)}{\psi_J(a)}=e^{\pm i2\vartheta}e^{\mp4\alpha\frac{l_a-l_J}{l_J}}e^{-\frac{2\alpha}{l_J}}$$

Of course, this ratios does not depend on b; therefore if the observable $\widetilde{\mathcal{O}}(b)=1$, then the dependence on b only appears in one correlation function through the cosine of the dihedral angle θ_a. As another example, we consider $\mathcal{O}(a)=l_a^{-1}$ and $\widetilde{\mathcal{O}}(b)=\frac{4l_J^2-l_b^2}{(2l_J)^4}$, then:

$$\begin{aligned}\langle\tfrac{\psi_J(a-1)}{\psi_J(a)}\tfrac{l_a-1/2}{l_a-1}\tfrac{4l_J^2-(l_a-1/2)^2}{4l_J^2}\tfrac{4l_J^2-l_b^2}{4l_J^2}\rangle_\psi\ &-2\langle\cos\theta_a\tfrac{4l_J^2-l_a^2}{4l_J^2}\tfrac{4l_J^2-l_b^2}{4l_J^2}+\tfrac{1}{16l_J^2}\tfrac{4l_J^2-l_b^2}{4l_J^2}\rangle_\psi\\ &+\langle\tfrac{\psi_J(a+1)}{\psi_J(l_a)}\tfrac{l_a+1/2}{l_a+1}\tfrac{(4l_J^2-(l_a+1/2)^2}{4l_J^2}\tfrac{4l_J^2-l_b^2}{4l_J^2}\rangle_\psi=0\end{aligned}$$

which can be approximated by:

$$\langle e^{-i2\vartheta}e^{4\alpha\frac{l_a-l_J}{l_J}}\Delta((l_a-1/2)^2)\Delta(l_b^2)\rangle_\psi-2e^{\frac{2\alpha}{l_J}}\langle\cos\theta_a\Delta(l_a^2)\Delta(l_b^2)+\frac{1}{16l_J^2}\Delta(l_b^2)\rangle_\psi+\langle e^{i2\vartheta}e^{-4\alpha\frac{l_a-l_J}{l_J}}\Delta((l_a+1/2)^2)\Delta(l_b^2)\rangle_\psi\approx0$$

where $\Delta(l_j^2)=\frac{l_j^2-4l_J^2}{4l_J^2}$.

B. Rescaling the Tetrahedron

We can now vary also the length scale l_J. First let's notice that in the same way we wrote an exact recursion relation for the leading order of the isosceles {6j}-symbol shifting the representation a (equation (5)), we can write a similar exact recursion relation for the leading order of the {6j}-symbol shifting the label J; that is

$$\sqrt{V_{J+1}(a,b)}\{a,b\}_{J+1}^{\rm LO}-2\cos(4\theta_J)\sqrt{V_J(a,b)}\{a,b\}_J^{\rm LO}+\sqrt{V_{J-1}(a,b)}\{a,b\}_{J-1}^{\rm LO}=0\tag{38}$$

Inserting this recursion relation on $J-$shifts in the definition correlation function, we obtain the following identity:

$$\begin{aligned}\langle\sqrt{V_{J+1}(a,b)}\frac{\psi_J(a)\psi_J(b)}{\psi_{J+1}(a)\psi_{J+1}(b)}\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_\psi \;\; &+\langle\sqrt{V_{J-1}(a,b)}\tfrac{\psi_J(a)\psi_J(b)}{\psi_{J-1}(a)\psi_{J-1}(b)}\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_\psi \\ &-2\langle\cos(4\theta_J)\sqrt{V_J(a,b)}\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_\psi \qquad =0\end{aligned} \tag{39}$$

The correlation functions appearing in this equation are in fact approximation. We are allowed to use the leading order of the {6j}-symbol because the boundary state used picks the function on large j_0. And for the same reason, we can expand $\sqrt{V_{J\pm1}(a,b)}$ and the ratios $\frac{\psi_J(a)\psi_J(b)}{\psi_{J\pm1}(a)\psi_{J\pm1}(b)}$:

$$\begin{aligned}\langle\sqrt{V_J(a,b)}\left(1-\tfrac{2l_J}{4l_J^2-l_a^2-l_b^2}\right)e^{-4\alpha\frac{(2l_J-(l_a+l_b))}{l_J}[1+\frac{3l_J-2(l_a+l_b)}{2l_J(2l_J-l_a-l_b)}]}\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_\psi - 2\langle\cos(4\theta_J)\sqrt{V_J(a,b)}\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_\psi \\ +\langle\sqrt{V_J(a,b)}\left(1+\tfrac{2l_J}{4l_J^2-l_a^2-l_b^2}\right)e^{4\alpha\frac{(2l_J-(l_a+l_b))}{l_J}[1-\frac{3l_J-2(l_a+l_b)}{2l_J(2l_J-l_a-l_b)}]}\mathcal{O}(a)\widetilde{\mathcal{O}}(b)\rangle_\psi \approx 0.\end{aligned} \tag{40}$$

We hope that such equation will turn out useful to study the asymptotic properties of the correlations function as the length scale J grows large, but we leave this for future investigation.

Conclusion

We have used the recursion relation satisfied by the {6j}-symbol to study the structure of its asymptotical expansion for large spins. The exact recursion relation allowed us to compute explicit the asymptotical approximation of the isosceles {6j}-symbol up to fourth order. This confirms previous results [11, 12] and introduces techniques allowing further systematic analytical calculations of the corrections to the behavior of the{6j}-symbol at large spins. However a clear and simple geometrical interpretation of the polynomials appearing in this expansion is still missing, but the differential equations that we provide for these coefficients should be a first step in this direction.

This work is useful in particular for the study of large scale correlations in the spinfoam model for 3d quantum gravity. In this context, the recursion relation allowed us to write equations satisfied by the spinfoam correlations similar to the Ward identities of standard quantum field theory. We hope that such recursion techniques can be further applied to the study of 4d spinfoam amplitudes and the resulting spinfoam graviton propagator [17].

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